quant-ph/9608008
LA-UR-96-2755

DISPLACEMENT-OPERATOR SQUEEZED STATES. I. TIME-DEPENDENT SYSTEMS HAVING ISOMORPHIC SYMMETRY ALGEBRAS

Michael Martin Nieto[1]
Theoretical Division, Los Alamos National Laboratory
University of California
Los Alamos, New Mexico 87545, U.S.A.

D. Rodney Truax[2]
Department of Chemistry
University of Calgary
Calgary, Alberta T2N 1N4, Canada

ABSTRACT

In this paper we use the Lie algebra of space-time symmetries to construct states which are solutions to the time-dependent Schrödinger equation for systems with potentials $V(x,\tau) = g^{(2)}(\tau)x^2 + g^{(1)}(\tau)x + g^{(0)}(\tau)$. We describe a set of number-operator eigenstates states, $\{\Psi_n(x,\tau)\}$, that form a complete set of states but which, however, are usually not energy eigenstates. From the extremal state, Ψ_0, and a displacement squeeze operator derived using the Lie symmetries, we construct squeezed states and compute expectation values for position and momentum as a function of time, τ. We prove a general expression for the uncertainty relation for position and momentum in terms of the squeezing parameters. Specific examples, all corresponding to choices of $V(x,\tau)$ and having isomorphic Lie algebras, will be dealt with in the following paper (II).

PACS: 03.65.-w, 02.20.+b, 42.50.-p

[1]Email: mmn@pion.lanl.gov
[2]Email: truax@acs.ucalgary.ca

1 Introduction

Recently [1], we have described the unsolved problem of how to define, for all systems, generalized squeezed states by the displacement-operator method. As a means to further elucidate this problem, we here undertake a study of systems where there is a Bogoliubov transformation, allowing displacement-operator squeezed states to be defined. These states can then be related to the ladder-operator squeezed states by this Bogoliubov transformation.

Specifically, we will discuss time-dependent systems which have isomorphic symmetry algebras. The isomorphism in the space-time symmetry algebras guarantees the existence of transformations which transform the time-dependent Schrödinger equation for all of these problems into a 'time-independent' Schrödinger equation for a one-dimensional harmonic oscillator. The presence of such a transformation means that the Bogoliubov transformation, discussed in Reference [1], exists and the displacement-operator squeezed states occur.

In the following paper (II) we explicitly construct squeezed states for special cases: the (well-known) harmonic oscillator, the free particle, the linear potential, the harmonic oscillator with a uniform driving force, and the repulsive oscillator.

In nonrelativistic quantum mechanics, time-dependent systems in one spatial dimension can be described by solutions to the time-dependent Schrödinger equation

$$\mathcal{S}_1 \Psi(x,\tau) = 0, \tag{1}$$

where the Schrödinger operator, $\mathcal{S}_1$, is

$$\mathcal{S}_1 = \partial_{xx} + 2i\partial_\tau - 2V(x,\tau). \tag{2}$$

The interaction, $V(x,\tau)$, that we will consider here has the form

$$V(x,\tau) = g^{(2)}(\tau)x^2 + g^{(1)}(\tau)x + g^{(0)}(\tau), \tag{3}$$

where the coefficients, $g^{(j)}(\tau)$, $j = 1, 2, 3$, are differentiable and piecewise continuous, but otherwise arbitrary. We denote the solution space of (1) by $\mathcal{F}_{S_1}$.

There are several common problems subsumed by the potential $V(x, \tau)$ in Eq. (3). We will discuss these individual cases in paper II. However, as we will be able to see in Section 2, all of these problems have isomorphic space-time symmetry algebras [2, 3, 4]. We will exploit this fact to algebraically calculate, in Section 3, states of the number operator for all such isomorphic systems.

These solution spaces are analogues of the number-operator states of the harmonic oscillator [3] and, in the case of the harmonic oscillator, they are indeed the usual number-operator states. In addition, for the harmonic oscillator they also correspond to the energy eigenstates. However, in general, for other potentials this will not be the case. Nevertheless, these solution spaces can be utilized in the calculation of properties of both coherent states [4] and squeezed states, for the general time-dependent potential (3) and also for the specific cases we will come to in paper II.

Coherent [5, 6, 7] and squeezed states [8]-[12] have received considerable attention in the literature in a number of contexts. In Section 4, we examine definitions of squeezed states in the light of the results of the Lie symmetry analysis of Section 2. In Sections 5, we calculate expectation values for position and momentum. We go on, in the next section, to obtain the uncertainties in position and momentum, and the uncertainty relation when $V(x, \tau)$ is given by Eq. (3).

2 Symmetry

The generators of space-time symmetries have the general form [2, 3],

$$\mathcal{L} = A(x, \tau)\partial_\tau + B(x, \tau)\partial_x + C(x, \tau). \tag{4}$$

For $\mathcal{L}$ to be a symmetry of Eq. (1), then $\mathcal{L}\Psi(x, \tau)$ must be a solution of Eq. (1) if $\Psi(x, \tau)$ is a solution. For this to be true, $\mathcal{L}$ must satisfy the equation [13]

$$[\mathcal{S}_1, \mathcal{L}] = \lambda(x, \tau)\mathcal{S}_1, \tag{5}$$

where λ is an as yet undetermined function of x and τ. The set of all such $\mathcal{L}$ form a Lie algebra, and the space-time Lie symmetry group is obtained accordingly [14].

The Lie group of space-time symmetries and its corresponding Lie algebra have been identified [2, 3] for systems with the interaction (3). The maximal, complex kinematical algebra is $su(1,1) \diamond w_1^c$. The generators of the space-time symmetries have the general form

$$
\begin{aligned}
\mathcal{J}_- &= \xi\partial_x - ix\dot{\xi} + i\mathcal{C}, && (6)\\
\mathcal{J}_+ &= \bar{\xi}\partial_x - ix\dot{\bar{\xi}} + i\bar{\mathcal{C}}, && (7)\\
I &= 1, && (8)\\
\mathcal{M}_- &= i[\phi_1\partial_\tau + (\tfrac{1}{2}\dot{\phi}_1 x + \mathcal{E}_1)\partial_x - \tfrac{i}{4}\ddot{\phi}_1 x^2 - ix\dot{\mathcal{E}}_1 + \tfrac{1}{4}\dot{\phi}_1 + i\mathcal{D}_1 + ig_0\phi_1], && (9)\\
\mathcal{M}_+ &= i[\phi_2\partial_\tau + (\tfrac{1}{2}\dot{\phi}_2 x + \mathcal{E}_2)\partial_x - \tfrac{i}{4}\ddot{\phi}_2 x^2 - ix\dot{\mathcal{E}}_2 + \tfrac{1}{4}\dot{\phi}_2 + i\mathcal{D}_2 + ig_0\phi_2], && (10)\\
\mathcal{M}_3 &= i[\phi_3\partial_\tau + (\tfrac{1}{2}\dot{\phi}_3 x + \mathcal{E}_3)\partial_x - \tfrac{i}{4}\ddot{\phi}_3 x^2 - ix\dot{\mathcal{E}}_3 + \tfrac{1}{4}\dot{\phi}_3 + i\mathcal{D}_3 + ig_0\phi_3]. && (11)
\end{aligned}
$$

The function ξ of τ and its complex conjugate $\bar{\xi}$ are constructed from two real solutions, χ_1 and χ_2, of the differential equation

$$\ddot{a} + 2g^{(2)}(\tau)a = 0. \tag{12}$$

We choose the Wronskian, $W(\chi_1,\chi_2) = \chi_1\dot{\chi}_2 - \dot{\chi}_1\chi_2 = 1$. The complex solutions of Eq. (12) are then,

$$\xi(\tau) = \tfrac{1}{\sqrt{2}}(\chi_1 + i\chi_2), \tag{13}$$

and its complex conjugate, $\bar{\xi}$. Their Wronskian is

$$W(\xi,\bar{\xi}) = \xi\dot{\bar{\xi}} - \dot{\xi}\bar{\xi} = -i. \tag{14}$$

We now define the remaining auxiliary τ-dependent functions. To begin,

$$\mathcal{C}(\tau) = \int^\tau d\rho\, \xi(\rho) g^{(1)}(\rho) = c(\tau) + \mathcal{C}^o, \tag{15}$$

where $\mathcal{C}^o$ is a complex integration constant and the function $c(\tau)$ is defined as

$$c(\tau) = \int_{\tau_o}^{\tau} d\rho\, \xi(\rho) g^{(1)}(\rho). \tag{16}$$

We shall take $\tau_o = 0$ from this point onward and in paper II. In addition, we have

$$\phi_1(\tau) = \xi^2, \quad \phi_2(\tau) = \bar{\xi}^2, \quad \phi_3(\tau) = 2\xi\bar{\xi}, \tag{17}$$

$$\begin{gathered} \mathcal{E}_1(\tau) = -\xi \left(i\mathcal{Q}_{1,2} + \mathcal{C} \right), \quad \mathcal{E}_2(\tau) = \bar{\xi} \left(i\mathcal{Q}_{2,1} - \bar{\mathcal{C}} \right), \\ \mathcal{E}_3(\tau) = \xi \left(i\mathcal{Q}_{2,1} - \bar{\mathcal{C}} \right) - \bar{\xi} \left(i\mathcal{Q}_{1,2} + \mathcal{C} \right), \end{gathered} \tag{18}$$

$$\begin{gathered} \mathcal{D}_1(\tau) = -\tfrac{1}{2} \left(i\mathcal{Q}_{1,2} + \mathcal{C} \right)^2, \quad \mathcal{D}_2(\tau) = -\tfrac{1}{2} \left(i\mathcal{Q}_{2,1} - \bar{\mathcal{C}} \right)^2, \\ \mathcal{D}_3(\tau) = \left(i\mathcal{Q}_{1,2} + \mathcal{C} \right) \left(i\mathcal{Q}_{2,1} - \bar{\mathcal{C}} \right), \end{gathered} \tag{19}$$

where

$$\begin{aligned} \mathcal{Q}_{1,2} &= \tfrac{1}{2} \xi^o \phi_3^o g^{(1)}(0) - q_{1,2} + i\mathcal{C}^o, \\ \mathcal{Q}_{2,1} &= \tfrac{1}{2} \bar{\xi}^o \phi_3^o g^{(1)}(0) - q_{2,1} + i\bar{\mathcal{C}}^o, \end{aligned} \tag{20}$$

and $q_{1,2}$ and $q_{2,1}$ are complex integration constants [15] such that $\bar{q}_{1,2} = q_{2,1}$ and $\bar{\mathcal{Q}}_{1,2} = \mathcal{Q}_{2,1}$.

Two integration constants appear in each of the equations (20). Without loss of generality, we can choose $q_{1,2}$ such that $\mathcal{Q}_{1,2} = 0$, and similarly, we select $q_{2,1}$ such that $\mathcal{Q}_{2,1} = 0$. The choices for the remaining integration constants, $\mathcal{C}^o$ and $\bar{\mathcal{C}}^o$, will be dictated by the physics of each individual system. Therefore, using Eqs. (17) through (19), we shall drop all references to $\mathcal{Q}_{1,2}$ and $\mathcal{Q}_{2,1}$. In addition, we define

$$\begin{gathered} \xi^o = \xi(0), \quad \bar{\xi}^o = \bar{\xi}(0), \quad \phi_3^o = \phi_3(0), \\ \dot{\xi}^o = \dot{\xi}(0), \quad \dot{\bar{\xi}}^o = \dot{\bar{\xi}}(0), \quad \dot{\phi}_3^o = \dot{\phi}_3(0). \end{gathered} \tag{21}$$

Note that ϕ_3^o is a real number since ϕ_3 is a real function of τ.

In the general case, calculations of expectation values are much simpler in terms of the complex algebra and the complex functions in Eqs. (13) and (15) through (19).

However, when working with actual examples, the real functions χ_1, χ_2, and the real counterparts of Eqs. (13) and (15) are more advantageous. In this paper, we shall use the complex functions. But in paper II, where we work with specific cases, we will transform all equations to expressions in terms of real functions.

The operators in Eqs. (6) through (11) satisfy the following (nonzero) commutation relations:

$$[\mathcal{J}_-, \mathcal{J}_+] = I, \tag{22}$$

$$[\mathcal{M}_+, \mathcal{M}_-] = -\mathcal{M}_3, \qquad [\mathcal{M}_3, \mathcal{M}_\pm] = \pm 2\mathcal{M}_\pm, \tag{23}$$

$$[\mathcal{M}_3, \mathcal{J}_-] = -\mathcal{J}_-, \qquad [\mathcal{M}_3, \mathcal{J}_+] = +\mathcal{J}_+, \tag{24}$$

$$[\mathcal{M}_-, \mathcal{J}_+] = -\mathcal{J}_-, \qquad [\mathcal{M}_+, \mathcal{J}_-] = +\mathcal{J}_+. \tag{25}$$

A number of formulae relating the τ-dependent functions in Eqs. (15) through (19) are proven in the Appendix. They are useful in establishing the commutation relations (23) to (25) as well as Eq. (26) below.

With these commutation relations, we see that the generators $\mathcal{J}_\pm$ and I form a complexified Heisenberg-Weyl algebra, w_1^c, and the operators $\mathcal{M}_3$ and $\mathcal{M}_\pm$ close under $su(1,1)$. Therefore, we have the Schrödinger algebra in one spatial dimension:

$$(\mathcal{SA})_1^c = su(1,1) \diamond w_1^c.$$

In the following section, we will restrict our analysis to a Lie subalgebra of $(\mathcal{SA})_1^c$ consisting of the operators $\mathcal{M}_3$, $\mathcal{J}_\pm$, and I. From the commutation relations in Eqs. (22) and (24), we recognize that these operators form a one-dimensional oscillator algebra, $os(1)$. It should be noted that the operator $\mathcal{J}_+$ is the Hermitian conjugate of $\mathcal{J}_-$. Also, I is clearly Hermitian. Lastly, the following identity can be demonstrated:

$$\mathcal{M}_3 = \tfrac{1}{2}\phi_3 \mathcal{S}_1 + \mathcal{J}_+\mathcal{J}_- + \tfrac{1}{2}, \tag{26}$$

This will prove useful in calculating the Casimir operator for $os(1)$.

3 Eigenstates of the Number Operator

Now we select the operators $\{\mathcal{M}_3, \mathcal{J}_\pm, I\}$ which satisfy the commutation relations

$$[\mathcal{M}_3, \mathcal{J}_\pm] = \pm\mathcal{J}_\pm, \quad [\mathcal{J}_-, \mathcal{J}_+] = I. \tag{27}$$

As mentioned above, we refer to this subalgebra of $(\mathcal{SA})_1^c$ as the oscillator subalgebra and denote it by $os(1)$. It has the Casimir operator

$$\mathbf{C} = \mathcal{J}_+\mathcal{J}_- - \mathcal{M}_3 = -\tfrac{1}{2}\phi_3\mathcal{S}_1 - \tfrac{1}{2}, \tag{28}$$

which commutes with all the generators in $os(1)$. The second equality in Eq. (28) follows from Eq. (26).

The fact that all the operators in $(\mathcal{SA})_1^c$ are constants of the motion on $\mathcal{F}_{S_1}$ follows from Eq. (5) [3]. We select two commuting constants of the motion, $\mathbf{C}$ and $\mathcal{M}_3$, and obtain a set of common eigenvectors. Also, we require that these eigenvectors satisfy the time-dependent Schrödinger equation (1) with potential (3). The $\mathcal{J}_\pm$ act as ladder operators on the eigenvalues of $\mathcal{M}_3$. There are three classes of irreducible representations of $os(1)$ [16]. We are only interested in the representation in which the spectrum of $\mathcal{M}_3$, $\mathrm{Sp}(\mathcal{M}_3)$, is bounded below. Therefore, we have the following [3]:

$$\mathcal{M}_3|m\rangle = (m + \tfrac{1}{2})|m\rangle, \quad \mathbf{C}|m\rangle = -\tfrac{1}{2}|m\rangle, \tag{29}$$

$$\mathcal{J}_+|m\rangle = \sqrt{m+1}|m+1\rangle, \quad \mathcal{J}_-|m\rangle = \sqrt{m}|m-1\rangle. \tag{30}$$

The condition that the spectrum of $\mathcal{M}_3$, $\mathrm{Sp}(\mathcal{M}_3)$, be bounded below is that

$$\mathcal{J}_-|0\rangle = 0, \tag{31}$$

which defines the extremal state for this representation space.

The states Ψ_m are called number-operator states because they are eigenfunctions of the number operator $\mathcal{J}_+\mathcal{J}_-$, where

$$\mathcal{J}_+\mathcal{J}_-|m\rangle = (\mathbf{C} + \mathcal{M}_3)|m\rangle = m|m\rangle. \tag{32}$$

(Note that we go back and forth between Ψ_m and the Dirac-Fock notation $|m\rangle$). It is important to keep in mind that the generators of $os(1)$ may involve an explicit time dependence. Furthermore, the eigenstates of $\mathcal{M}_3$ are solutions to the time-dependent Schrödinger equation and are not eigenstates of the Hamiltonian except, as we shall see, in the case of the harmonic oscillator. Therefore, the number-operator states are not generally energy eigenstates. However, they do provide a convenient, complete basis for our purposes [4].

From Eqs. (29) and (31), we can calculate the specific form of the wave functions. From the first equation in (29), we obtain a first-order partial differential equation for Ψ_m which can be integrated by the method of characteristics [2, 15, 17]. This method leads to $\mathcal{R}$-separation of variables [13] and yields

$$\Psi_m(x,\tau) = \exp\{i\mathcal{R}(x,\tau)\}\psi_m(\zeta)\Xi_m(\eta), \tag{33}$$

where the $\mathcal{R}$-factor is

$$\mathcal{R}(x,\tau) = \tfrac{1}{4}\frac{x^2}{\phi_3}\left(\dot{\phi}_3 - \dot{\phi}_3^o\right) + \frac{x}{\phi_3^{1/2}}\left(\frac{\mathcal{E}_3}{\phi_3^{1/2}} - \frac{\mathcal{E}_3^o}{(\phi_3^o)^{1/2}} + \tfrac{1}{2}\mathcal{B}_3\phi_3^o\right), \tag{34}$$

and the $\mathcal{R}$-separable coordinates are

$$\zeta = \frac{x}{\phi_3^{1/2}} - \mathcal{B}_3, \quad \eta = \tau. \tag{35}$$

The η-dependent function, Ξ_m, is

$$\Xi_m(\eta) = \left(\frac{\phi_3^o}{\phi_3}\right)^{\frac{1}{4}}\left(\frac{\xi^o\bar{\xi}(\eta)}{\bar{\xi}^o\xi(\eta)}\right)^{\frac{1}{2}\left(m+\frac{1}{2}\right)}\exp\left[-i\left(\Lambda_3(\eta) + G^{(0)}(\eta)\right)\right], \tag{36}$$

where $\mathcal{E}_3^o = \mathcal{E}_3(0)$ from Eq. (18) is a real constant. The real number ϕ_3^o is given in Eq. (21). Furthermore, $\mathcal{B}_3(\tau)$ is defined by the first equality

$$\mathcal{B}_3(\tau) = \int_0^\tau ds\,\frac{\mathcal{E}_3(s)}{\phi_3^{3/2}(s)} = b_3(\tau) - b_3(0), \tag{37}$$

where

$$b_3(\tau) = \frac{1}{\phi_3^{1/2}}\left[i\left(\xi\bar{\mathcal{C}} - \bar{\xi}\mathcal{C}\right)\right], \tag{38}$$

is a real function of τ. See the Appendix (Formula I) for a proof of the second equality in Eq. (37). In addition, we define

$$G^{(0)}(\tau) = \int_0^\tau ds\, g^{(0)}(s), \tag{39}$$

and

$$\Lambda_3(\tau) = \int_0^\tau ds \left(\frac{\mathcal{E}_3^2(s)}{\phi_3^2(s)} + \frac{\mathcal{D}_3(s)}{\phi_3(s)} \right) - \mathcal{B}_3 \left(\frac{\mathcal{E}_3^o}{(\phi_3^o)^{1/2}} + \tfrac{1}{4}\mathcal{B}_3^2 \dot{\phi}_3^o \right). \tag{40}$$

Applying $\mathcal{J}_-$ to Ψ_0 from Eq. (33) produces a first-order ordinary differential equation in ζ for ψ_0. Solving this equation leads to a normalized extremal-state wave function of the form

$$\begin{aligned}\Psi_0(x,\tau) &= \left(\frac{1}{\pi\phi_3^o}\right)^{\frac{1}{4}} \exp\left(-b_3^2(0)/2\right) \exp\{i\mathcal{R}\} \exp\left[-(1-i\theta_1)\zeta^2 + (b_3(0)+i\theta_2)\zeta\right] \\ &\quad \times \left(\frac{\phi_3^o}{\phi_3}\right)^{\frac{1}{4}} \left(\frac{\xi^o\bar{\xi}(\eta)}{\bar{\xi}^o\xi(\eta)}\right)^{\frac{1}{4}} \exp\left[-i\left(\Lambda_3(\eta) + G^{(0)}(\eta)\right)\right],\end{aligned} \tag{41}$$

where

$$\theta_1 = \tfrac{1}{2}\dot{\phi}_3^o, \qquad \theta_2 = \frac{\mathcal{E}_3^o}{(\phi_3^o)^{1/2}}. \tag{42}$$

The wave function for the state with quantum number m has the form

$$\begin{aligned}\Psi_m(x,\tau) &= \left(\frac{1}{m!}\right)^{\frac{1}{2}} \left(\frac{1}{2}\right)^{\frac{m}{2}} \left(\frac{\bar{\xi}^o}{\xi^o}\right)^{\frac{1}{4}} \exp\{i\mathcal{R}(x,\tau)\}\psi_m(\zeta) \\ &\quad \times \left(\frac{\phi_3^o}{\phi_3}\right)^{\frac{1}{4}} \left(\frac{\bar{\xi}}{\xi}\right)^{\frac{1}{2}\left(m+\frac{1}{2}\right)} \exp\left[-i\left(\Lambda_3(\eta) + G^{(0)}(\eta)\right)\right],\end{aligned} \tag{43}$$

where

$$\psi_m(\zeta) = H_m(\zeta - b_3(0)) \left(\frac{1}{\pi\phi_3^o}\right)^{\frac{1}{4}} \exp\left[-\tfrac{1}{2}(1-i\theta_1)\zeta^2 + (b_3(\tau_o)+i\theta_2)\zeta\right] \tag{44}$$

and $H_m(\zeta - b_3(0))$ is Hermite polynomial given by the Rodrigues formula

$$H_m(\zeta - b_3(0)) = (-)^m \exp\left[\zeta^2 - 2b_3(0)\zeta\right] \partial_\zeta^m \exp\left[-\zeta^2 + b_3(0)\zeta\right]. \tag{45}$$

4 Coherent and Squeezed States

In notation modified for the present problem, we review the general formalism for displacement-operator states.

4.1 Coherent states

The displacement-operator coherent states [18, 19], Ψ_α, for the systems described by the Schrödinger equation (2) and (3), are defined by

$$|\alpha\rangle = D(\alpha)|0\rangle, \tag{46}$$

where α is a complex number and the displacement operator

$$D(\alpha) = \exp\left(\alpha \mathcal{J}_+ - \bar{\alpha}\mathcal{J}_-\right), \tag{47}$$

is unitary. The state Ψ_0 is the extremal state (41) in the number-operator basis, discussed in the previous section. Computationally, a more convenient form for the displacement operator is given by the expression

$$D(\alpha) = \exp\left(-\tfrac{1}{2}|\alpha|^2\right)\exp\left(\alpha \mathcal{J}_+\right)\exp\left(-\bar{\alpha}\mathcal{J}_-\right). \tag{48}$$

4.2 Squeezed states

The generalized squeezed state $|\alpha, z\rangle$ can be obtained from

$$|\alpha, z\rangle = D(\alpha)S(z)|0\rangle, \tag{49}$$

where z is a complex parameter and $S(z)$, the squeeze operator, is

$$S(z) = \exp\left(z\mathcal{K}_+ - \bar{z}\mathcal{K}_-\right). \tag{50}$$

The state $|0\rangle$ is the extremal number-operator state (41). The operators $\mathcal{K}_\pm$ and $\mathcal{K}_3$ are

$$\mathcal{K}_- = \tfrac{1}{2}\mathcal{J}_-^2, \quad \mathcal{K}_+ = \tfrac{1}{2}\mathcal{J}_+^2, \quad \mathcal{K}_3 = \mathcal{J}_+\mathcal{J}_- + \tfrac{1}{2}. \tag{51}$$

These three operators satisfy an $su(1,1)$ Lie algebra with commutation relations

$$[\mathcal{K}_+, \mathcal{K}_-] = -\mathcal{K}_3, \qquad [\mathcal{K}_3, \mathcal{K}_\pm] = \pm\mathcal{K}_\pm. \tag{52}$$

Notice the difference between the definition of K_0 in reference [1] and the operator, $\mathcal{K}_3$, defined above. We have

$$\mathcal{K}_3 = 2K_0, \qquad \mathcal{K}_\pm = K_\pm. \tag{53}$$

This difference is reflected in the commutation relations above but does not affect the remaining calculations in any way. The operators $\mathcal{K}_-$, $\mathcal{K}_+$, and $\mathcal{K}_3$ have the important properties

$$(\mathcal{K}_-)^\dagger = \mathcal{K}_+, \quad (\mathcal{K}_+)^\dagger = \mathcal{K}_-, \quad (\mathcal{K}_3)^\dagger = \mathcal{K}_3, \tag{54}$$

that is, the operators $\mathcal{K}_-$ and $\mathcal{K}_+$ are Hermitian conjugates while $\mathcal{K}_3$ is Hermitian. Therefore, the squeeze operator $S(z)$ is unitary.

The commutation relations of the $\mathcal{K}_\pm$ and $\mathcal{K}_3$ with $\mathcal{J}_\pm$ are

$$\begin{aligned} &[\mathcal{K}_-, \mathcal{J}_-] = 0, \quad [\mathcal{K}_+, \mathcal{J}_-] = -\mathcal{J}_+, \quad [\mathcal{K}_3, \mathcal{J}_-] = -\mathcal{J}_-, \\ &[\mathcal{K}_-, \mathcal{J}_+] = \mathcal{J}_-, \quad [\mathcal{K}_+, \mathcal{J}_+] = 0, \quad [\mathcal{K}_3, \mathcal{J}_+] = +\mathcal{J}_+. \end{aligned} \tag{55}$$

We can express $S(z)$ more conveniently through the Baker-Campbell-Hausdorff [9, 20] relations as

$$S(z) = \exp(\gamma_+\mathcal{K}_+)\exp(\gamma_3\mathcal{K}_3)\exp(\gamma_-\mathcal{K}_-), \tag{56}$$

where γ_-, γ_+, and γ_3 are analytic functions of z and $\bar{z}$

$$\begin{aligned} \gamma_- = -\tfrac{\bar{z}}{|z|}\tanh|z|, \quad \gamma_+ = \tfrac{z}{|z|}\tanh|z|, \\ \gamma_3 = -\ln(\cosh|z|). \end{aligned} \tag{57}$$

The analytical mappings $\gamma_\pm$ and γ_3 are referred to as canonical coordinates of the second kind. Most of our calculations will be carried out with canonical coordinates of the second kind.

A definition of squeezed states that is different than Eq. (49) can be given by

$$|z,\alpha\rangle = S(z)D(\alpha)|0\rangle. \tag{58}$$

We refer to the squeezed state in Eq. (49) as the (α, z)-representation and to that in (58) as the (z, α)-representation. The order of the parameters z and α indicates the order the two operators $S(z)$ and $D(\alpha)$ have been applied to the extremal state.

Although explicit knowledge of the squeezed-state wave functions is not necessary for computation of expectation values of functions of position and momentum, it is often important to have some representation for them. One approach is to write them as expansions in terms of eigenstates of the number operator. According to Eq. (41), the extremal state is a Gaussian function. Starting with the definition (49) and the operators (48) for $D(\alpha)$ and (56) for $S(z)$, we have

$$|\alpha,z\rangle = e^{-\frac{1}{2}|\alpha|^2} e^{\alpha\mathcal{J}_+} e^{-\bar{\alpha}\mathcal{J}_-} e^{\gamma_+\mathcal{K}_+} e^{\gamma_3\mathcal{K}_3} e^{\gamma_-\mathcal{K}_-}|0\rangle. \tag{59}$$

Given Eq. (31), the definition (51), and the fact that $\mathcal{K}_3|0\rangle = (1/2)|0\rangle$, we obtain

$$|\alpha,z\rangle = e^{\frac{1}{2}(\gamma_3-|\alpha|^2)} e^{\alpha\mathcal{J}_+} e^{-\bar{\alpha}\mathcal{J}_-} e^{\gamma_+\mathcal{K}_+}|0\rangle. \tag{60}$$

Next, using the relationship

$$e^{-\bar{\alpha}\mathcal{J}_-} e^{\gamma_+\mathcal{K}_+} = e^{(\gamma_+\mathcal{K}_+ - \gamma_+\bar{\alpha}\mathcal{J}_+)} e^{-\bar{\alpha}\mathcal{J}_-}, \tag{61}$$

and since $[\mathcal{K}_+, \mathcal{J}_+] = 0$, we find that

$$|\alpha,z\rangle = e^{\frac{1}{2}(\gamma_3-|\alpha|^2)} e^{\gamma_+\mathcal{K}_+} e^{(\alpha-\gamma_+\bar{\alpha})\mathcal{J}_+}|0\rangle. \tag{62}$$

Expanding the exponentials about the identity, noting Eq. (30), and using $|m\rangle = \sqrt{(1/m!)}\mathcal{J}_+^m|0\rangle$, we get double summations in terms of the odd and even eigenstates

$$
\begin{aligned}
|\alpha, z\rangle &= e^{\frac{1}{2}(\gamma_3-|\alpha|^2)} \\
&\times\left\{\sum_{m=0}^{\infty}\left[\sum_{n=0}^{m}\sqrt{\frac{(2m)!}{(2n)!}}\frac{(\alpha-\gamma_+\bar{\alpha})^{2n}\gamma_+^{m-n}}{2^{m-n}(m-n)!}\right]|2m\rangle\right. \\
&\left.+\sum_{m=0}^{\infty}\left[\sum_{n=0}^{m}\sqrt{\frac{(2m+1)!}{(2n+1)!}}\frac{(\alpha-\gamma_+\bar{\alpha})^{2n+1}\gamma_+^{m-n}}{2^{m-n}(m-n)!}\right]|2m+1\rangle\right\},
\end{aligned}
\tag{63}
$$

where $\gamma_\pm$ and γ_3 are given by Eq. (57). We can derive an expression for $|z,\alpha\rangle$ in a similar manner, obtaining

$$
\begin{aligned}
|z, \alpha\rangle &= e^{\frac{1}{2}(\gamma_3+\alpha^2\gamma_- -|\alpha|^2)} \\
&\times\left\{\sum_{m=0}^{\infty}\left[\sum_{n=0}^{m}\sqrt{\frac{(2m)!}{(2n)!}}\frac{\alpha^{2n}e^{2n\gamma_3}\gamma_+^{m-n}}{2^{m-n}(m-n)!}\right]|2m\rangle\right. \\
&\left.+\sum_{m=0}^{\infty}\left[\sum_{n=0}^{m}\sqrt{\frac{(2m+1)!}{(2n+1)!}}\frac{\alpha^{2n+1}e^{(2n+1)\gamma_3}\gamma_+^{m-n}}{2^{m-n}(m-n)!}\right]|2m+1\rangle\right\}.
\end{aligned}
\tag{64}
$$

We shall compare expectation values for the two representations of squeezed states in the next section.

5 Expectation Values for Squeezed States

In this section we calculate the expectation values of position and momentum in both the (α, z)- and the (z, α)-representations for potentials of the type (3), where we now use the definitions

$$
\alpha = |\alpha|e^{i\delta}, \quad z = re^{i\theta}, \quad r = |z|. \tag{65}
$$

We will derive the phase-space trajectories for systems with the general potential (3).

Note that

$$
x = \bar{\xi}\mathcal{J}_- + \xi\mathcal{J}_+ + i\left(\xi\bar{\mathcal{C}} - \bar{\xi}\mathcal{C}\right), \tag{66}
$$

$$
p = \dot{\bar{\xi}}\mathcal{J}_- + \dot{\xi}\mathcal{J}_+ + i\left(\dot{\xi}\bar{\mathcal{C}} - \dot{\bar{\xi}}\mathcal{C}\right). \tag{67}
$$

The proof of Eqs. (66) and (67) is easily demonstrated. We need only the Wronskian (14) and the definitions (6) of $\mathcal{J}_-$ and (7) of $\mathcal{J}_+$. We see that

$$
\begin{aligned}
\bar{\xi}\mathcal{J}_- + \xi\mathcal{J}_+ &= x - i\left(\xi\bar{\mathcal{C}} - \bar{\xi}\mathcal{C}\right), \\
\dot{\bar{\xi}}\mathcal{J}_- + \dot{\xi}\mathcal{J}_+ &= -i\partial_x - i\left(\dot{\xi}\bar{\mathcal{C}} - \dot{\bar{\xi}}\mathcal{C}\right). \qquad (68)
\end{aligned}
$$

By rearranging Eqs. (68), we obtain Eqs. (66) and (67).

We compute the expectation values in both the (α, z)- and (z, α)-representations. Let $\mathcal{O}$ be an operator. Then, we have the expectation value $\langle\mathcal{O}\rangle$ of $\mathcal{O}$ in each of the representations

$$
\begin{aligned}
\langle\mathcal{O}\rangle_{(\alpha,z)} &= \langle\alpha, z|\mathcal{O}|\alpha, z\rangle, \\
&= \langle 0|S^{-1}(z)D^{-1}(\alpha)\mathcal{O}D(\alpha)S(z)|0\rangle, \qquad (69) \\
\langle\mathcal{O}\rangle_{(z,\alpha)} &= \langle z, \alpha|\mathcal{O}|z, \alpha\rangle. \\
&= \langle 0|D^{-1}(\alpha)S^{-1}(z)\mathcal{O}S(z)D(\alpha)|0\rangle. \qquad (70)
\end{aligned}
$$

For position and momentum operators in the (α, z)-representation, we have

$$
S^{-1}(z)D^{-1}(\alpha)xD(\alpha)S(z) = X_-(\tau)\mathcal{J}_- + X_+(\tau)\mathcal{J}_+ + X_0(\tau)I, \qquad (71)
$$

$$
S^{-1}(z)D^{-1}(\alpha)pD(\alpha)S(z) = \dot{X}_-(\tau)\mathcal{J}_- + \dot{X}_+(\tau)\mathcal{J}_+ + \dot{X}_0(\tau)I, \qquad (72)
$$

where we define the coefficients

$$\begin{aligned}
X_-(\tau) &= \bar{\xi}(e^{\gamma_3} - \gamma_-\gamma_+ e^{-\gamma_3}) - \xi\gamma_- e^{-\gamma_3}, \\
&= \bar{\xi}\cosh r + \xi\frac{\bar{z}}{r}\sinh r, \\
&= \bar{\xi}\cosh r + \xi e^{-i\theta}\sinh r,
\end{aligned} \tag{73}$$

$$\begin{aligned}
X_+(\tau) &= \xi e^{-\gamma_3} + \bar{\xi}\gamma_+ e^{-\gamma_3}, \\
&= \xi\cosh r + \bar{\xi}\frac{z}{r}\sinh r, \\
&= \xi\cosh r + \bar{\xi}e^{i\theta}\sinh r,
\end{aligned} \tag{74}$$

$$\begin{aligned}
X_0(\tau) &= \alpha\bar{\xi} + \bar{\alpha}\xi + i\left(\xi\bar{\mathcal{C}} - \bar{\xi}\mathcal{C}\right), \\
&= |\alpha|[e^{i\delta}\bar{\xi} + e^{-i\delta}\xi] + i\left(\xi\bar{\mathcal{C}} - \bar{\xi}\mathcal{C}\right),
\end{aligned} \tag{75}$$

and we have used Eqs. (57) and (65). In the (z, α)-representation, we find that

$$D^{-1}(\alpha)S^{-1}(z)x(\tau)S(z)D(\alpha) = X_-(\tau)\mathcal{J}_- + X_+(\tau)\mathcal{J}_+ + Y_0(\tau)I, \tag{76}$$

$$D^{-1}(\alpha)S^{-1}(z)p(\tau)S(z)D(\alpha) = \dot{X}_-(\tau)\mathcal{J}_- + \dot{X}_+(\tau)\mathcal{J}_+ + \dot{Y}_0(\tau)I, \tag{77}$$

where the coefficient $Y_0(\tau)$ is

$$\begin{aligned}
Y_0(\tau) &= \alpha X_- + \bar{\alpha}X_+ + i\left(\xi\bar{\mathcal{C}} - \bar{\xi}\mathcal{C}\right), \\
&= \left(\alpha\bar{\xi} + \bar{\alpha}\xi\right)\cosh r + \frac{(\alpha\bar{z}\xi + \bar{\alpha}z\bar{\xi})}{r}\sinh r \\
&\quad + i\left(\xi\bar{\mathcal{C}} - \bar{\xi}\mathcal{C}\right), \\
&= |\alpha|[(\bar{\xi}e^{i\delta} + \xi e^{-i\delta})\cosh r + (\bar{\xi}e^{i(\theta-\delta)} + \xi e^{-i(\theta-\delta)})\sinh r] \\
&\quad + i\left(\xi\bar{\mathcal{C}} - \bar{\xi}\mathcal{C}\right),
\end{aligned} \tag{78}$$

and we have used Eq. (65) in the last identity.

Since we have $\langle 0|J_-|0\rangle = \langle 0|J_+|0\rangle = 0$, we find that the expectation value for position in the (α, z)-representation is

$$\langle x(\tau)\rangle_{(\alpha,z)} = X_0, \tag{79}$$

where X_0 is given by Eq. (75). The expectation value for momentum in this representation is

$$
\begin{aligned}
\langle p(\tau)\rangle_{(\alpha,z)} &= \dot{X}_0, \\
&= \alpha\dot{\bar{\xi}} + \bar{\alpha}\dot{\xi} + i\left(\dot{\xi}\bar{\mathcal{C}} - \dot{\bar{\xi}}\mathcal{C}\right), \\
&= |\alpha|[e^{i\delta}\dot{\bar{\xi}} + e^{-i\delta}\dot{\xi}] + i\left(\dot{\xi}\bar{\mathcal{C}} - \dot{\bar{\xi}}\mathcal{C}\right).
\end{aligned} \tag{80}
$$

At time $\tau = 0$, let x_o and p_o be the initial position and momentum, respectively. Then, we have

$$
\begin{aligned}
\langle x(0)\rangle_{(\alpha,z)} = x_o = \alpha\bar{\xi}^o + \bar{\alpha}\xi^o + i\left(\xi^o\bar{\mathcal{C}}^o - \bar{\xi}^o\mathcal{C}^o\right), \\
\langle p(0)\rangle_{(\alpha,z)} = p_o = \alpha\dot{\bar{\xi}}^o + \bar{\alpha}\dot{\xi}^o + i\left(\dot{\xi}^o\bar{\mathcal{C}}^o - \dot{\bar{\xi}}\mathcal{C}^o\right).
\end{aligned} \tag{81}
$$

By making use of the Wronskian at $\tau = \tau_o$,

$$
\alpha = i\left(p_o\xi^o - x_o\dot{\xi}^o\right) + i\mathcal{C}^o. \tag{82}
$$

Substituting for α and $\bar{\alpha}$ in (79) and (80), we get the general expressions

$$
\begin{aligned}
\langle x(\tau)\rangle_{(\alpha,z)} &= i\{[\bar{\xi}(\tau)\xi^o - \xi(\tau)\bar{\xi}^o]p_o + [\xi(\tau)\dot{\bar{\xi}}^o - \bar{\xi}(\tau)\dot{\xi}^o]x_o\} \\
&\quad + i\left(\xi(\tau)\bar{c}(\tau) - \bar{\xi}(\tau)c(\tau)\right),
\end{aligned} \tag{83}
$$

$$
\begin{aligned}
\langle p(\tau)\rangle_{(\alpha,z)} &= i\{[\dot{\bar{\xi}}(\tau)\xi^o - \dot{\xi}(\tau)\bar{\xi}^o]p_o + [\dot{\xi}(\tau)\dot{\bar{\xi}}^o - \dot{\bar{\xi}}(\tau)\dot{\xi}^o]\} \\
&\quad + i\left(\dot{\xi}(\tau)\bar{c}(\tau) - \dot{\bar{\xi}}(\tau)c(\tau)\right),
\end{aligned} \tag{84}
$$

where $c(\tau)$ is defined by Eq. (16).

The expectation values in the (z,α)-representation are calculated in a similar way. For position, we have

$$
\langle x(\tau)\rangle_{(z,\alpha)} = Y_0, \tag{85}
$$

where Y_0 is given in Eq. (78). For momentum, we obtain

$$\begin{aligned}
\langle p(\tau)\rangle_{(z,\alpha)} &= \dot{Y}_0 && (86)\\
&= \left(\alpha\dot{\bar{\xi}}+\bar{\alpha}\dot{\xi}\right)\cosh r+\frac{\alpha\bar{z}\dot{\xi}+\bar{\alpha}z\dot{\bar{\xi}}}{r}\sinh r \\
&\qquad +i\left(\dot{\xi}\bar{\mathcal{C}}-\dot{\bar{\xi}}\mathcal{C}\right), \\
&= |\alpha|\left[\left(e^{i\delta}\dot{\bar{\xi}}+e^{-i\delta}\dot{\xi}\right)\cosh r+\left(e^{i(\delta-\theta)}\dot{\xi}+e^{-i(\delta-\theta)}\dot{\bar{\xi}}\right)\sinh r\right] \\
&\qquad +i\left(\dot{\xi}\bar{\mathcal{C}}-\dot{\bar{\xi}}\mathcal{C}\right). && (87)
\end{aligned}$$

From the initial conditions, we get the relationships

$$|\alpha|[e^{i\delta}\cosh r+e^{-i(\delta-\theta)}\sinh r]=i(p_o\xi^o-x_o\dot{\xi}^o)+i\mathcal{C}^o, \tag{88}$$

and its complex conjugate. When these equations are substituted into Eqs. (85) and (87), we obtain results which are identical to Eqs. (83) and (84), respectively, in the (α,z)-representation. Since the expectation values of position and momentum are identical in both the (z,α)- and (α,z)-representations, when we write the expectation values of position and momentum in terms of the initial position and momentum, we will now drop the representation labels in Eqs. (83) and (84).

6 Uncertainty Products for Squeezed States

Next we want to evaluate Heisenberg uncertainty product, $(\Delta x)(\Delta p)$, where

$$(\Delta x)^2=\langle x^2(\tau)\rangle-\langle x(\tau)\rangle^2,\quad (\Delta p)^2=\langle p^2(\tau)\rangle-\langle p(\tau)\rangle^2. \tag{89}$$

In the (α,z)-representation the uncertainty in position (89) can be calculated using (66) and (79):

$$(\Delta x)^2_{(\alpha,z)}=X_+X_-+X_0^2-X_0^2=X_+X_-, \tag{90}$$

where X_- and X_+ are given by Eqs. (73) and (74), respectively. In the (z,α)-representation we find the same result, since

$$(\Delta x)^2_{(z,\alpha)}=X_+X_-+Y_0^2-Y_0^2=X_+X_-, \tag{91}$$

where we have employed Eq. (85). Because Eqs. (90) and (91) are identical, we simply write

$$(\Delta x)^2 = X_+ X_-, \tag{92}$$

$$= \xi\bar{\xi}\cosh 2r + \tfrac{1}{2}(\bar{\xi}^2 e^{i\theta} + \xi^2 e^{-i\theta})\sinh 2r, \tag{93}$$

where we have made use of Eqs. (73) and (74). Similarly, we find that the uncertainty in momentum is independent of the representation, and we obtain

$$(\Delta p)^2 = \dot{X}_+ \dot{X}_-, \tag{94}$$

$$= \dot{\xi}\dot{\bar{\xi}}\cosh 2r + \tfrac{1}{2}(\dot{\bar{\xi}}^2 e^{i\theta} + \dot{\xi}^2 e^{-i\theta})\sinh 2r. \tag{95}$$

Therefore, in either representation, the uncertainty relation in position and momentum is

$$(\Delta x)^2(\Delta p)^2 = X_+ X_- \dot{X}_+ \dot{X}_-. \tag{96}$$

Substituting for X_- and X_+, we have

$$\begin{aligned}(\Delta x)^2(\Delta p)^2 &= \xi\bar{\xi}\dot{\xi}\dot{\bar{\xi}}\cosh^2 2r \\ &+ \tfrac{1}{4}(\bar{\xi}^2 e^{i\theta} + \xi^2 e^{-i\theta})(\dot{\bar{\xi}}^2 e^{i\theta} + \dot{\xi}^2 e^{-i\theta})\sinh^2 2r \\ &- \tfrac{1}{2}[\bar{\xi}\dot{\bar{\xi}}(\xi\dot{\bar{\xi}} + \dot{\xi}\bar{\xi})e^{i\theta} + \xi\dot{\xi}(\xi\dot{\bar{\xi}} + \dot{\xi}\bar{\xi})e^{-i\theta}]\cosh 2r\sinh 2r,\end{aligned} \tag{97}$$

in terms of the complex functions.

Finally, replacing ξ and $\bar{\xi}$ by Eq. (13), we obtain an expression for the uncertainty product in terms of the real functions, χ_1 and χ_2. This result,

$$\begin{aligned}(\Delta x)^2(\Delta p)^2 &= \tfrac{1}{4}[1 + (\chi_1\dot{\chi}_1 + \chi_2\dot{\chi}_2)^2] + \tfrac{1}{8}\Big\{[1 + 3(\chi_1\dot{\chi}_1 + \chi_2\dot{\chi}_2)^2] \\ &+[(\chi_1\dot{\chi}_1 - \chi_2\dot{\chi}_2)^2 - (\chi_1\dot{\chi}_2 + \dot{\chi}_1\chi_2)^2]\cos 2\theta \\ &\quad + 2(\chi_1\dot{\chi}_1 - \chi_2\dot{\chi}_2)(\chi_1\dot{\chi}_2 + \dot{\chi}_1\chi_2)\sin 2\theta\Big\}\sinh^2 2r \\ &\quad - \tfrac{1}{4}(\chi_1\dot{\chi}_1 + \chi_2\dot{\chi}_2)[(\chi_1\dot{\chi}_1 - \chi_2\dot{\chi}_2)\cos\theta \\ &\quad + (\chi_1\dot{\chi}_2 + \dot{\chi}_1\chi_2)\sin\theta]\sinh 4r,\end{aligned} \tag{98}$$

is more revealing and prepares us for paper II. (Notice that when $z = 0$, then expression (97) or (98) reduces to the usual uncertainty product for coherent states [4].)

Acknowledgements

MMN acknowledges the support of the United States Department of Energy. DRT acknowledges a grant from the Natural Sciences and Engineering Research Council of Canada.

Appendix

In this Appendix, we prove four formulae which interrelate time-dependent auxiliary functions. We refer the reader to Section II for the definitions of the special functions required in the proofs. The first three formulas are helpful for calculating the commutation relations of Eqs. (23) to (26). Formula IV derives Eq. (37).

Formula I.

$$\tfrac{1}{2}\dot{\phi}_3 b_3(\tau) + \frac{\mathcal{E}_3}{\phi_3^{1/2}} = i\phi_3^{1/2}(\dot{\xi}\bar{\mathcal{C}} - \dot{\bar{\xi}}\mathcal{C}). \tag{99}$$

Proof: Using the definitions of ϕ_3 and b_3, we have

$$\tfrac{1}{2}\dot{\phi}_3 b_3 + \frac{\mathcal{E}_3}{\phi_3^{1/2}} = \tfrac{1}{2}\dot{\phi}_3 b_3 + \phi_3 \dot{b}_3. \tag{100}$$

From Eq. (109), we see that

$$\dot{b}_3 = -\tfrac{1}{2}\frac{\dot{\phi}_3}{\phi_3} b_3 + \frac{i}{\phi_3^{1/2}}. \tag{101}$$

Multiplying by ϕ_3 and rearranging, we obtain

$$\tfrac{1}{2}\dot{\phi}_3 b_3 + \phi_3 \dot{b}_3 = \frac{i}{\phi_3^{1/2}}, \tag{102}$$

and we are done.

Formula II.

$$\frac{\ddot{\phi}_3}{\phi_3} - \tfrac{1}{2}\frac{\dot{\phi}_3^2}{\phi_3^2} = -4g_2 + \frac{2}{\phi_3^2}. \tag{103}$$

Proof: Substituting for ϕ_3, we obtain

$$\frac{\ddot{\phi}_3}{\phi_3} - \tfrac{1}{2}\frac{\dot{\phi}_3^2}{\phi_3^2} = 2\frac{(\ddot{\xi}\bar{\xi} + 2\dot{\xi}\dot{\bar{\xi}})}{\phi_3} - 2\frac{(\dot{\xi}\bar{\xi} + \xi\dot{\bar{\xi}})^2}{\phi_3^2}, \tag{104}$$

$$= \frac{-8g_2\xi\bar{\xi}}{\phi_3} + 4\frac{\dot{\xi}\dot{\bar{\xi}}}{\phi_3} - 2\frac{(\dot{\xi}\bar{\xi} + \xi\dot{\bar{\xi}})^2}{\phi_3^2}, \tag{105}$$

$$= -4g_2 - 2\frac{(\xi\dot{\bar{\xi}} - \dot{\xi}\bar{\xi})^2}{\phi_3^2}. \tag{106}$$

To obtain Eq. (105) from Eq. (104), we used the differential equation (12) for the solutions ξ and $\bar{\xi}$. Then using the Wronskian (14) in Eq. (106), we get (103).

Formula III.

$$2\frac{\dot{\mathcal{E}}_3}{\phi_3} - \frac{\dot{\phi}_3\mathcal{E}_3}{\phi_3^2} = -2g_1 - \frac{2}{\phi_3^{3/2}}b_3. \tag{107}$$

Proof: Substituting the definitions for $\mathcal{E}_3$ and ϕ_3, we observe that

$$2\frac{\dot{\mathcal{E}}_3}{\phi_3} - \frac{\dot{\phi}_3\mathcal{E}_3}{\phi_3^2} = -2\frac{(\dot{\xi}\bar{\mathcal{C}} + \dot{\bar{\xi}}\mathcal{C} + 2g_1\xi\bar{\xi})}{\phi_3} + 2\frac{(\dot{\xi}\bar{\xi} + \xi\dot{\bar{\xi}})(\xi\bar{\mathcal{C}} + \bar{\xi}\mathcal{C})}{\phi_3^2},$$

$$= -2g_1 - \frac{2}{\phi_2^2}[-(\xi\dot{\bar{\xi}} - \dot{\xi}\bar{\xi})\xi\bar{\mathcal{C}} + (\xi\dot{\bar{\xi}} - \dot{\xi}\bar{\xi})\bar{\xi}\mathcal{C}],$$

$$= -2g_1 - \frac{2i}{\phi_3^2}(\xi\bar{\mathcal{C}} - \bar{\xi}\mathcal{C}). \tag{108}$$

When we combine Eq. (109) with (108), we obtain the desired result.

Formula IV, Eq. (37).

$$\mathcal{B}_3(\tau) = b_3(\tau) - b_3(0), \tag{109}$$

where

$$b_3(\tau) = \frac{i(\xi\bar{\mathcal{C}} - \bar{\xi}\mathcal{C})}{\phi_3^{1/2}}. \tag{110}$$

Proof: Recall that we have chosen $\mathcal{Q}_{1,2} = \mathcal{Q}_{2,1} = 0$. From the definitions of $\mathcal{E}_3$ and $\mathcal{B}_3$ in Eqs. (18) and (37), respectively, we have

$$\mathcal{B}_3 = \int_0^\tau ds\, \frac{\mathcal{E}_3}{\phi_3^{3/2}} = -\int_0^\tau ds\, \frac{\xi\bar{\mathcal{C}}}{\phi_3^{3/2}} - \int_\tau ds\, \frac{\bar{\xi}\mathcal{C}}{\phi_3^{3/2}}. \tag{111}$$

Inserting the Wronskian and the definition of ϕ_3 yields the result

$$\begin{aligned}
\mathcal{B}_3 &= -\tfrac{i}{2^{3/2}} \int_0^\tau ds\, \bar{\mathcal{C}}(s) \frac{\xi(s)\left(\xi(s)\dot{\bar{\xi}}(s) - \dot{\xi}(s)\bar{\xi}(s)\right)}{\xi(s)^{3/2}\bar{\xi}^{3/2}} \\
&\quad - \tfrac{i}{2^{3/2}} \int_0^\tau ds\, \mathcal{C}(s) \frac{\bar{\xi}(s)\left(\xi(s)\dot{\bar{\xi}}(s) - \dot{\xi}(s)\bar{\xi}(s)\right)}{\xi(s)^{3/2}\bar{\xi}^{3/2}}, \\
&= \tfrac{i}{2^{1/2}} \int_0^\tau ds\, \bar{\mathcal{C}}(s)\, d\left(\frac{\xi^{1/2}(s)}{\bar{\xi}^{1/2}(s)}\right) - \tfrac{i}{2^{1/2}} \int_0^\tau ds\, \mathcal{C}(s)\, d\left(\frac{\bar{\xi}^{1/2}(s)}{\xi^{1/2}(s)}\right).
\end{aligned} \tag{112}$$

Integrating by parts, we have

$$\begin{aligned}
\mathcal{B}_3(\tau) &= \tfrac{i}{2^{1/2}} \bar{\mathcal{C}}(s) \left(\frac{\xi^{1/2}(s)}{\bar{\xi}^{1/2}(s)}\right)\Bigg|_0^\tau - \tfrac{i}{2^{1/2}} \mathcal{C}(s) \left(\frac{\bar{\xi}^{1/2}(s)}{\xi^{1/2}(s)}\right)\Bigg|_0^\tau, \\
&= \tfrac{i}{2^{1/2}} \left(\bar{\mathcal{C}}(\tau)\frac{\xi^{1/2}(\tau)}{\bar{\xi}^{1/2}(\tau)} - \bar{\mathcal{C}}^o \frac{(\xi^o)^{1/2}}{(\bar{\xi}^o)^{1/2}}\right) \\
&\quad - \tfrac{i}{2^{1/2}} \left(\mathcal{C}(\tau)\frac{\bar{\xi}^{1/2}(\tau)}{\xi^{1/2}(\tau)} - \mathcal{C}^o \frac{(\bar{\xi}^o)^{1/2}}{(\xi^o)^{1/2}}\right).
\end{aligned} \tag{113}$$

Rearranging this expression, we get

$$\mathcal{B}_3(\tau) = \frac{i}{\phi_3^{1/2}} \left(\xi(\tau)\bar{\mathcal{C}}(\tau) - \bar{\xi}(\tau)\mathcal{C}(\tau)\right) - \frac{i}{(\phi_3^o)^{1/2}} \left(\xi^o\bar{\mathcal{C}}^o - \bar{\xi}^o\mathcal{C}^o\right), \tag{114}$$

which is just Eq. (109) given the definition of $b_3(\tau)$ in Eq. (110).

References

[1] M. M. Nieto and D. R. Truax, Forschritte der Physik, (in press).

[2] D. R. Truax, J. Math. Phys. **22**, 1959 (1981).

[3] D. R. Truax, J. Math. Phys. **23**, 43 (1982).

[4] S. Gee and D.R. Truax, Phys. Rev. A **29**, 1627 (1984).

[5] J. R. Klauder and B.-S. Skagerstam, *Coherent States – Applications in Physics and Mathematical Physics* (World Scientific, Singapore, 1985).

[6] M. M. Nieto and L. M. Simmons, Jr., Phys. Rev. Lett. **41**, 207 (1978); Phys. Rev. D **20**, 1321, 1332 (1979); the first two of a series concluding with M. M. Nieto, L. M. Simmons, Jr., and V. P. Gutschick, Phys. Rev. D **23**, 927 (1981); M. M. Nieto, in *Coherent States – Applications in Physics and Mathematical Physics*.

[7] G. Schrade, V. I. Man'ko, W. P. Schleich, and R. J. Glauber, Quantum Semiclass. Opt. **7**, 307 (1995).

[8] H. P. Yuen, Phys. Rev. A **13**, 2226 (1976); J. N. Hollenhorst, Phys. Rev. D **19**, 1669 (1979); *Workshop on Squeezed States and Uncertainty Relations*, edited by D. Han, Y.S. Kim, and W. W. Zachary, NASA Conference Publication No. 3135 (NASA, Washington, DC, 1992).

[9] R. A. Fisher, M. M. Nieto, and V. D. Sandberg, Phys. Rev. D**29**, 1107 (1984).

[10] M. M. Nieto, in *Frontiers of Nonequilibrium Statistical Physics*, edited by G. T. Moore and M. O. Scully (Plenum, New York, 1986).

[11] M. M. Nieto and D. R. Truax, Phys. Rev. Lett. **71**, 2843 (1994).

[12] M. M. Nieto, Quantum Opt. **6**, 9 (1994). The large round brackets in Eq. (5) should be squared. See. Eq. (4.11) of Ref. [10].

[13] W. Miller Jr., *Symmetry and Separation of Variables*. (Addison-Wesley, Reading, Mass., 1977).

[14] W. Miller Jr., *Symmetry Groups and their Applications*. (Academic, New York, 1972).

[15] A. Kalivoda and D. R. Truax, to be published.

[16] W. Miller Jr., *Lie Theory and Special Functions*. (Academic, New York, 1968).

[17] E. C. Zachmanoglou and D. W. Thoe, *Introduction to Partial Differential Equations with Applications*. (Williams and Wilkins, Baltimore, MD, 1976).

[18] J. R. Klauder, J. Math. Phys. **4**, 1058 (1963).

[19] A. M. Perelomov, Commun. Math. Phys. **26**, 222 (1972).

[20] D. R. Truax, Phys. Rev. D **31**, 1988 (1985).